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Abstract: There are two types $i = 1, 2$ of particles on the line R, with N_i particles of type i. Each particle of type i moves with constant velocity v_i. Moreover, any particle of type $i = 1, 2$ jumps to any particle of type $j = 1, 2$ with rates $N_j^{-1}\alpha_{ij}$. We find phase transitions in the clusterization (synchronization) behaviour of this system of particles on different time scales $t = t(N)$ relative to $N = N_1 + N_2$.

Key-words: time synchronization, particle systems, asymptotitcs, martingale problem, recurrent equations

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Modèles de Synchronisation de Temps

Résumé : On considère une modèle de synchronisation de temps pour N processeurs.

Mots-clés : systèmes de particules, temps, synchronisation, processeur

1 The Model

In general, time synchronization problem can be presented as follows. There are N systems (processors, units, persons etc.) There is an absolute (physical) time t, but each processor j fulfills a homogeneous job in its own proper time $t_j = w_j t, w_j > 0$. Proper time is measured by the amount w_j of the job, accomplished by the processor for the unit of the physical time, if it is disjoint from other processors. However, there is a communication between each pair of processors, which should lead to drastic change of their proper times.

There can be many variants of exact formulation of such problem, see [3, 6, 9]. We will call the model considered here the basic model, because there are no restrictions on the jump process. Many other problems include such restrictions, for example, only jumps to the left are allowed. Due to absence of restrictions, this problem, as we will see below, is a "linear problem" in the sense that after scalings it leads to linear equations. In despite of this it has nontrivial behaviour, one sees different picture on different time scales.

We give now equivalent exact formulation of the model we consider here, in terms of particles. On the real line there are N_1 particles of type 1 and N_2 particles of type 2 correspondingly, $N = N_1 + N_2$. Each particle of type $i = 1, 2$ performs two independent movements. First of all, it moves with constant speed v_i in the positive direction. We assume further that v_i are constant and different, thus we can assume without loss of generality that $0 \leq v_1 < v_2$. The degenerate case $v_1 = v_2$ is different and will be considered separately.

Secondly, at any time interval $[t, t + dt]$ each particle of type i independently of the others with probability $\alpha_{ij} dt$ decides to make a jump to some particle of type j and chooses the coordinate of the j-type particle, where to jump, among the particles of type j, with probability $\frac{1}{N_j}$. Here α_{ij} are given nonnegative parameters for $i, j = 1, 2$. Further on, unless otherwise stated, we assume that $\alpha_{11} = \alpha_{22} = 0, \alpha_{12}, \alpha_{21} > 0$.

After such instantaneous jump the particle of type i continues the movement with the same velocity v_i. This defines continuous time Markov chain $\left\{x_k^{(i)}(t)\right\}, i = 1, 2; k = 1, ..., N_i$, where $x_k^{(i)}(t)$ is the coordinate of k-th particle of type i at time t. We assume that the initial coordinates $x_k^{(i)}(0)$ of the particles at time 0 are given. Coordinates $x_k^{(i)}(t)$ can be interpreted as the modified proper times of the particles, the nonmodified proper time being $x_k^{(i)}(0) + v_i t$.

2 Main results

We show that the process consists of three consecutive stages: initial desynchronization up to the critical scale, critical slow down of desynchronization and final stabilization.

Final stabilization The first theorem shows that for N_i fixed and $t \to \infty$ there is a synchronization: all particles asymptotically, as $t \to \infty$, move with the same constant velocity v, that is like vt. However it does not say how fluctuations depend on N_i.

Put

$$m(t) = \min_{i,k} x_k^{(i)}(t)$$

Theorem 1 *For any fixed* N_1, N_2 *there exists* $v = v(N_1, N_2) > 0$ *such that for any* $i = 1, 2$ *and any* $k = 1, ..., N_i$ *a.s.*

$$\lim_{t\to\infty} \frac{x_k^{(i)}(t)}{t} = v$$

Moreover, the distribution of the vector $\left\{x_k^{(i)}(t) - m(t), i = 1, 2; k = 1, ..., N_i\right\}$ *tends to a stationary distribution.*

The velocity v will be written down explicitely in terms of this distribution, it depends of course on α_{ij} and v_i. Note that both the velocity and the distribution do not depend on the initial coordinates.

Initial desynchronization Now we consider the case when $N \to \infty$ but t is fixed. More exactly, we consider a sequence of pairs (N_1, N_2) such that $N_1, N_2 \to \infty$ so that $\frac{N_i}{N} \to c_i$, where $c_1 + c_2 = 1, c_i > 0$. It is convenient here to consider positive measures or generalized functions

$$m^{(N_i)}(t, x) = \frac{1}{N_i} \sum_k \delta(x - x_k^{(i)}(t)), x \in R_+$$

defined by the coordinates of N_i particles of type i at time t. We assume that at time $t = 0$ for any bounded C^1-functions $\phi_i(x)$ on R the sequence $< m_i^{(N_i)}(0, .), \phi_i >$ converges to some number.

Theorem 2 *Then for any* t *there are weak deterministic limits*

$$\lim_{N\to\infty} \frac{1}{N} m_i^{(N_i)}(t, x) = m_i(t, x)$$

where $m_i(t, x)$ *satisfy the following equations*

$$\frac{\partial m_1}{\partial t} + v_1 \frac{\partial m_1}{\partial x} = \alpha_{12}(m_2 - m_1) \tag{1}$$

$$\frac{\partial m_2}{\partial t} + v_2 \frac{\partial m_2}{\partial x} = \alpha_{21}(m_1 - m_2) \tag{2}$$

Now we want to study the asymptotic behaviour of $m_i(x, t)$ for $t \to \infty$. Denote

$$a_i(t) = \int x m_i(x, t) dx, \qquad d_i(t) = \int (x - a_i(t))^2 m_i(t, x)\, dx$$

Theorem 3 *There exist constants $v, d > 0$ such that as $t \to \infty$*

$$a_i(t) = vt + a_{i0} + o(1)$$
$$d_i(t) = dt + d_{i0} + o(1)$$

for some constants a_{i0}, d_{i0}. Moreover,

$$\Delta_i(x,t) = \frac{m_i(x,t) - a_i(t)}{\sqrt{d_i(t)}}$$

tends to $\frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})$ pointwise as $t \to \infty$.

Critical point and uniform estimates Here we assume that $N_1 = [c_1 N]$, $N_2 = [c_2 N]$ for some $c_i > 0, c_1 + c_2 = 1$. Introduce the empirical means (mass centres) for types 1 and 2

$$\overline{x^{(i)}}(t) = \frac{1}{N_i} \sum_{k=1}^{N_i} x_k^{(i)}(t),$$

the empirical variances

$$S_i^2(t) = \frac{1}{N_i} \sum_{k=1}^{N_i} \left(x_k^{(i)}(t) - \overline{x^{(i)}}(t) \right)^2$$

and their means

$$\boldsymbol{\mu}_i(t) = \mathsf{E}\overline{x^{(i)}}(t), \quad \boldsymbol{l}_{12}(t) = \boldsymbol{\mu}_1(t) - \boldsymbol{\mu}_2(t), \quad R_i(t) = \mathsf{E}S_i^2(t)$$

The following asymptotic results hold for any sequence of pairs (N, t) with $N \to \infty$ and $t = t(N) \to \infty$.

Theorem 4 *In probability*

$$\boldsymbol{l}_{12}(t) \to \frac{v_1 - v_2}{\alpha_{12} + \alpha_{21}}, \qquad \frac{\boldsymbol{\mu}_i(t)}{t} \to \frac{\alpha_{12} v_2 + \alpha_{21} v_1}{\alpha_{12} + \alpha_{21}}$$

Assume now that $N_i = c_i N$, where $c_i > 0, c_1 + c_2 = 1$.

Theorem 5 *There are the following three regions of asymptotic behaviour, uniform in $t(N)$ for sufficiently large N:*

- *if $\frac{t(N)}{N} \to 0$, then $R_i(t(N)) \sim h\varkappa_2 t(N)$,*
- *if $t = t(N) = sN$ for some $s > 0$, then $R_i(t(N)) \sim h\,(1 - e^{-\varkappa_2 s})N$,*
- *if $\frac{t(N)}{N} \to \infty$, then $R_i(t(N)) \sim hN$,*

where the constant $\varkappa_2 > 0$ is defined by the formula (21), see below, and

$$h = \frac{2\alpha_{12}\alpha_{21}\,(v_1 - v_2)^2}{\varkappa_2(\alpha_{12} + \alpha_{21})^3}.$$

3 Limit $t \to \infty$

In this section we will prove Theorem 1.

Two particles. It is useful to consider first the case when $N_1 = N_2 = 1$. Thus consider the process $(x^{(1)}(t), x^{(2)}(t))$. We will prove that there exist deterministic limits

$$\lim_{t\to\infty} \frac{x^{(i)}(t)}{t} = v$$

for $i = 1, 2$ and some $v > 0$, moreover the distribution of the random variable $\rho(t) = x^{(2)}(t) - x^{(1)}(t)$ tends to some distribution on R_+.

We can assume that $v_1 = 0, v_2 > 0$. The Markov chain $\rho(t) = x^{(2)}(t) - x^{(1)}(t)$ on R_+ satisfies the Doeblin condition, that is from any $x \in R_+$ there is a jump rate to 0, bounded away from zero, here it equals $\alpha_{12} + \alpha_{21}$. It follows that $\rho(t)$ is ergodic. Then as $t \to \infty$ there exists the limiting (invariant) distribution $F(x)$ for $\rho(t)$. Let

$$t_1 < t_2 < ...$$

time moments when $x^{(1)}(t) = x^{(2)}(t)$. It is clear that $t_k - t_{k-1}$ are independent random variables, exponentially distributed with parameter $\alpha_{12}+\alpha_{21}$. It follows that $F(x)$ is exponential with the density

$$p(x) = \lambda \exp(-\lambda x), \quad \lambda = \frac{\alpha_{12} + \alpha_{21}}{v_2 - v_1}$$

Thus, if the limits $\lim_{t\to\infty} \frac{x_i(t)}{t}$ exist, then they are equal. Let us prove that they exist and

$$v = v_1 + \alpha_{12} \int xp(x)dx \tag{3}$$

In fact, the particle 1 moves with constant speed v_1 and performs on the time interval $[0, T]$ independent exponentially distributed jumps in the positive direction. As $T \to \infty$, the number of these jumps asymptotically equals $\alpha_{12}T$, and the mean jump asymptotically is $\int xp(x)dx$.

Similarly one can get

$$v = v_2 - \alpha_{21} \int xp(x)dx \tag{4}$$

From this and (3) we have

$$v = \frac{\alpha_{21}v_1 + \alpha_{12}v_2}{\alpha_{21} + \alpha_{12}}$$

General case. Let us prove first the second statement of the theorem. We can put $v_1 = 0$ and change the coordinate system putting $m(t) = 0$. Consider a configuration of particles at time t. Denote the particle, which has coordinate $m(t) = 0$ at time t, as particle 0. Let

$p(t+2)$ be the probability that at time $t+2$ each particle will be inside the interval $[0, 2v_2]$. This probability can be (very roughly) estimated from below as

$$p(t+2) \geq \min(p_{01}p_2p_1, p_{02}p_3p_4)$$

To prove this consider first the case when the particle 0 has type 1. Under this condition $p(t+2)$ can be estimated from below as $p_{01}p_2p_1$, where p_{01} is the probability that particle 0 does not do any jumps in the time interval $(t, t+2)$, p_2 is the probability that each particle of type 2 jumps at least once to the particle 0 in the time interval $(t, t+1)$ and does not do any more jumps in the time interval $(t, t+2)$, p_1 is the probability that each particle of type 1 jumps to some particle of type 2 in the time interval $(t+1, t+2)$. Similarly, under the condition that the particle 0 has type 2, $p(t+2)$ can be estimated from below as $p_{02}p_3p_4$, where p_{02} is the probability that the particle 0 does not do any jumps in the time interval $(t, t+2)$, p_3 is the probability that each particle of type 1 jumps at least once to the particle 0 in the time interval $(t, t+1)$ and does not do any more jumps in the time interval $(t, t+2)$, p_4 is the probability that each particle of type 2 jumps to some particle of type 1 in the time interval $(t+1, t+2)$.

This means that the Markov chain $\mathcal{L} = \left\{ x_k^{(i)}(t) - m(t), i = 1, 2; k = 1, ..., N_i \right\}$ satisfies the Doeblin condition. Then it is ergodic and has some stationary distribution. We will write now formula for v, assuming however that $\alpha_{ii} = 0$. For this we need some marginals of this stationary distribution.

Let $A_i(t)$ be the event that at time t at the point $m(t)$ there is a particle of type i, and $q_i = \lim_{t\to\infty} P(A_i(t))$ be the stationary (limiting) probability of A_i. Let $p_i(y)$ be the stationary conditional (under the condition A_i) probability density of the distance from m to the nearest particle. In the time interval $[T, T+dt]$ the particle in $m(t)$ moves with the speed v_i, and moreover can make one jump. This gives, for example under the condition A_1, constant movement $v_1 dt$ of m, and the jump of m to the nearest point with rate $\alpha_{12}dt$. Thus as $T \to \infty$ we have

$$\mathsf{E}(m(T+dt) \,|\, m(T)\,) - m(T) \;=\; q_1(v_1 + \alpha_{12} \int y p_1(y)dy)dt + q_2(v_2 + \alpha_{21} \int y p_2(y)dy)dt + o(1)$$

and then

$$v = q_1(v_1 + \alpha_{12} \int y p_1(y)dy) + q_2(v_2 + \alpha_{21} \int y p_2(y)dy)$$

About Doeblin chains. In the standard theory of Doeblin chains, see [2], it is assumed that transition probabilities are absolutely continuous with respect to some positive measure μ on the state space.

If at time 0 all $x_k^{(i)}$ are different, then for any t it is true that all $x_k^{(i)}$ are different a.s. Thus transition probabilities (for example, for the embedded chain at times $0, 1, 2, ...$) are absolutely continuous with respect to Lebesgue measure on $(R_+^{N_1-1} \times R_+^{N_2}) \cup (R_+^{N_1} \times R_+^{N_2-1})$. If at time 0 some coordinates coincide, then a.s. in finite time τ they become all different.

4 Limit $N \to \infty$

It is very intuitive to introduce the following continuous model. Let $m_i(0,x), x \in R$, $i = 1,2$, be positive smooth functions, $M_i = \int m_i(0,x)dx = 1$. We call them continuous mass distributions of type i at time $t = 0$. The dynamics of the masses is deterministic - during time dt from each element dm_1 of the mass the part $\alpha_{12} dt dm_1$ goes out and distributes correspondingly to the mass $m_2(x)$, namely it becomes the mass distribution with density $m_2(x)\alpha_{12} dt dm_1$, and vice-versa, interchanging 1 and 2. Moreover each mass element moves with velocities v_1 and v_2 correspondingly. From this we easily get linear equations (1)–(2) for mass distribution $m_i(t,x)$ at time t with the initial conditions

$$m_i(0,x) = f_i(x)$$

Now we will prove convergence of N particle model to the continuous model.

4.1 Convergence: the martingale problem

Here we prove Theorem 2.

We consider continuous time Markov process

$$\xi_{N_1,N_2}(t) = \left(x_1^{(1)}(t), \ldots, x_{N_1}^{(1)}(t); x_1^{(2)}(t), \ldots, x_{N_2}^{(2)}(t)\right) \tag{5}$$

with the state space $R^{N_1+N_2}$. Its generator

$$\begin{aligned}
\left(L_{N_1,N_2} f\right)\left(x^{(1)}; x^{(2)}\right) &= \left[v_1 \sum_{i=1}^{N_1} \frac{\partial}{\partial x_i^{(1)}} + v_2 \sum_{j=1}^{N_2} \frac{\partial}{\partial x_j^{(2)}}\right] f\left(x^{(1)}; x^{(2)}\right) + \\
&+ \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \frac{\alpha_{12}}{N_2} \left[f\left(\left(x^{(1)}; x^{(2)}\right)_{i \to j}\right) - f\left(x^{(1)}; x^{(2)}\right)\right] + \\
&+ \sum_{j=1}^{N_2} \sum_{i=1}^{N_1} \frac{\alpha_{21}}{N_1} \left[f\left(\left(x^{(1)}; x^{(2)}\right)_{i \leftarrow j}\right) - f\left(x^{(1)}; x^{(2)}\right)\right],
\end{aligned}$$

where the following notation is used

$$\begin{aligned}
\left(x^{(1)}; x^{(2)}\right) &= \left(x_1^{(1)}, \ldots, x_{N_1}^{(1)}; x_1^{(2)}, \ldots, x_{N_2}^{(2)}\right), \\
\left(x^{(1)}; x^{(2)}\right)_{i \to j} &= \left(x_1^{(1)}, \ldots, x_{i-1}^{(1)}, x_j^{(2)}, x_{i+1}^{(1)}, \ldots, x_{N_1}^{(1)}; x_1^{(2)}, \ldots, x_{N_2}^{(2)}\right), \\
\left(x^{(1)}; x^{(2)}\right)_{i \leftarrow j} &= \left(x_1^{(1)}, \ldots, x_{N_1}^{(1)}; x_1^{(2)}, \ldots, x_{j-1}^{(2)}, x_i^{(1)}, x_{j+1}^{(2)}, \ldots, x_{N_2}^{(2)}\right),
\end{aligned}$$

is defined on bounded C^1-functions.

We will consider the limiting behaviour of this process when $t = const$, $N_1, N_2 \to \infty$. It is not convenient to deal with the sequence $\xi_{N_1,N_2}(t)$ of processes because the dimension of the state space changes with N_1, N_2.

Denote

$$M_{N_1,N_2}(t) = \left(\frac{1}{N_1} \sum_{i=1}^{N_1} \delta(\cdot - x_i^{(1)}(t)), \frac{1}{N_2} \sum_{j=1}^{N_2} \delta(\cdot - x_j^{(2)}(t)) \right).$$

where $\delta(x)$, $x \in R$, is the δ-function. One can see that the generalized functions

$$\frac{1}{N_1} \sum_{i=1}^{N_1} \delta(\cdot - x_i^{(1)}(t)), \qquad \frac{1}{N_2} \sum_{j=1}^{N_2} \delta(\cdot - x_j^{(2)}(t))$$

represent empirical "densities" or masses of (type 1 and 2 correspondingly) particles at time t. Thus, if $\phi(x) = (\phi_1(x), \phi_2(x))$, where $\phi_i \in S(R)$, then for fixed $x_1^{(1)}(t), \ldots, x_{N_1}^{(1)}(t)$, $x_1^{(2)}(t), \ldots, x_{N_2}^{(2)}(t)$ the vector function $M_{N_1,N_2}(t)$ is a linear functional on the vector test functions ϕ, that is

$$\langle M_{N_1,N_2}(t), \phi \rangle = \frac{1}{N_1} \sum_{i=1}^{N_1} \phi_1(x_i^{(1)}(t)) + \frac{1}{N_2} \sum_{j=1}^{N_2} \phi_2(x_j^{(2)}(t)).$$

Fix some $T > 0$. Then $(M_{N_1,N_2}(t),\ 0 \le t \le T)$ can be considered as a Markov process taking its values in the space of tempered distributions $S'(R) \times S'(R)$. In the sequel we consider $S'(R) \times S'(R)$ as a topological space equipped with the strong topology (see Subsection 6.2). Without loss of generality one can assume that the trajectories of the process $M_{N_1,N_2}(t)$ are right continuous functions with left limits. So it is natural to consider the Skorohod space $\Pi^T = D([0,T], S'(R) \times S'(R))$ of functions on $[0,T]$ with values in $S'(R) \times S'(R)$ as a coordinate space of the process $M_{N_1,N_2}(t)$. Subsection 6.2 explains how to introduce topology on this space. Let $\mathcal{B}(\Pi^T)$ be the corresponding Borel σ-algebra. Denote $P^T_{N_1,N_2}$ the probability measure on $\left(\Pi^T, \mathcal{B}(\Pi^T)\right)$, induced by the process $(M_{N_1,N_2}(t),\ 0 \le t \le T)$.

Our assumption for the theorem is that for any test function $\phi(x)$ the sequence $\langle M_{N_1,N_2}(0), \phi \rangle$ weakly converges as $N_1, N_2 \to \infty$.

We want to prove that as $N_1, N_2 \to \infty$ the sequence of probability distributions $P^T_{N_1,N_2}$ has a weak limit, and this limit is a one-point measure, that is the only trajectory $(m_1(t), m_2(t))$, $0 \le t \le T$, which is the classical solution of the system (1)-(2). We split a proof of this result into the next two propositions.

Proposition 6 *The family of probability distributions* $\left\{P^T_{N_1,N_2}\right\}_{N_1,N_2}$ *on* $\left(\Pi^T, \mathcal{B}(\Pi^T)\right)$ *is tight.*

Proposition 7 *Limit points of the family of distributions* $P^T_{N_1,N_2}$ *are concentrated on the weak solutions of the system (1)-(2).*

4.1.1 Tightness

Before proving Proposition 6 we start with some preliminary lemmas. We want to prove that the family of distributions $P^T_{N_1,N_2}$ of the random process $(M_{N_1,N_2}(t),\, 0 \le t \le T)$, with values in the space of generalized functions, is tight. By the theorem 4.1 of [8] (see also Subsection 6.2), it is sufficient to prove that for any test function $\psi = (\psi_1(x), \psi_2(x))$ the family of random processes $(\langle M_{N_1,N_2}(t), \psi\rangle\,,\, 0 \le t \le T)$, with values in R^1, is tight. This will be done in the Proposition 10 below.

Fix some test function $\psi = (\psi_1(x), \psi_2(x))$ and consider the random process

$$\begin{aligned} F_{\psi,N_1,N_2}\left(x^{(1)}(t); x^{(2)}(t)\right) &= \langle M_{N_1,N_2}(t), \psi\rangle = \\ &= \frac{1}{N_1}\sum_{i=1}^{N_1} \psi_1(x_i^{(1)}(t)) + \frac{1}{N_2}\sum_{j=1}^{N_2} \psi_2(x_j^{(2)}(t)) \end{aligned}$$

This is a function of the Markov process $\xi_{N_1,N_2}(t)$, thus (see [5, Lemma 5.1, p. 330], for example) the following two processes are martingales:

$$\begin{aligned} W_{\psi,N_1,N_2}(t) &= F_{\psi,N_1,N_2}\left(x^{(1)}(t); x^{(2)}(t)\right) - F_{\psi,N_1,N_2}\left(x^{(1)}(0); x^{(2)}(0)\right) - \\ &\quad - \int_0^t L_{N_1,N_2} F_{\psi,N_1,N_2}\left(x^{(1)}(s); x^{(2)}(s)\right) ds \qquad (6) \\ V_{\psi,N_1,N_2}(t) &= \left(W_{\psi,N_1,N_2}(t)\right)^2 - \int_0^t L_{N_1,N_2} F^2_{\psi,N_1,N_2}\left(x^{(1)}(s); x^{(2)}(s)\right) ds + \\ &\quad + 2\int_0^t F_{\psi,N_1,N_2}\left(x^{(1)}(s); x^{(2)}(s)\right) L_{N_1,N_2} F_{\psi,N_1,N_2}\left(x^{(1)}(s); x^{(2)}(s)\right) ds. \end{aligned}$$

For shortness we will write $F(x^{(1)}; x^{(2)})$ instead of $F_{\psi,N_1,N_2}(x^{(1)}; x^{(2)})$.

Lemma 8 *The following estimates hold:*

i) $\left|L_{N_1,N_2} F\left(x^{(1)}; x^{(2)}\right)\right| \le C_1(\psi, v_1, v_2, \alpha_{12}, \alpha_{21})$ *uniformly in* N_1, N_2 *and* $\left(x^{(1)}; x^{(2)}\right)$*;*

ii) *uniformly in* $x^{(1)}$, $x^{(2)}$

$$\left|L_{N_1,N_2} F^2(x^{(1)}; x^{(2)}) - F(x^{(1)}; x^{(2)}) L_{N_1,N_2} F(x^{(1)}; x^{(2)})\right| \le \frac{C_{12}(\alpha_{12}, \psi_1)}{N_1} + \frac{C_{21}(\alpha_{21}, \psi_2)}{N_2}. \qquad (7)$$

Proof of the lemma. Note that

$$\begin{aligned} F\left(\left(x^{(1)}; x^{(2)}\right)_{i\to j}\right) - F\left(x^{(1)}; x^{(2)}\right) &= \frac{1}{N_1}\left(\psi_1\left(x_j^{(2)}\right) - \psi_1\left(x_i^{(1)}\right)\right), \\ F\left(\left(x^{(1)}; x^{(2)}\right)_{i\leftarrow j}\right) - F\left(x^{(1)}; x^{(2)}\right) &= \frac{1}{N_2}\left(\psi_2\left(x_i^{(1)}\right) - \psi_2\left(x_j^{(2)}\right)\right). \end{aligned}$$

Thus

$$\begin{aligned} L_{N_1,N_2}F\left(x^{(1)};x^{(2)}\right) &= \frac{v_1}{N_1}\sum_{i=1}^{N_1}\psi_1'\left(x_i^{(1)}\right)+\frac{v_2}{N_2}\sum_{j=1}^{N_2}\psi_2'\left(x_j^{(2)}\right)+ \\ &+\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{\alpha_{12}}{N_2}\cdot\frac{1}{N_1}\left(\psi_1\left(x_j^{(2)}\right)-\psi_1\left(x_i^{(1)}\right)\right)+ \\ &+\sum_{j=1}^{N_2}\sum_{i=1}^{N_1}\frac{\alpha_{21}}{N_1}\cdot\frac{1}{N_2}\left(\psi_2\left(x_i^{(1)}\right)-\psi_2\left(x_j^{(2)}\right)\right). \end{aligned} \tag{8}$$

Then

$$\begin{aligned} \left|L_{N_1,N_2}F\left(x^{(1)};x^{(2)}\right)\right| &\leq |v_1|\,\|\psi_1'\|_C+|v_2|\,\|\psi_2'\|_C+ \\ &+2\alpha_{12}\,\|\psi_1\|_C+2\alpha_{21}\,\|\psi_2\|_C \end{aligned}$$

and the assertion **i)** of the lemma is proved. To prove assertion **ii)** it is convenient to represent $L_{N_1,N_2}=L^0_{N_1,N_2}+L^1_{N_1,N_2}$ as the sum of "differential" $L^0_{N_1,N_2}$and "jump" $L^1_{N_1,N_2}$ parts.

It is easy to see that

$$L^0_{N_1,N_2}F^2(x^{(1)};x^{(2)})-2F(x^{(1)};x^{(2)})L^0_{N_1,N_2}F(x^{(1)};x^{(2)})=0.$$

Let us prove that uniformly in $x^{(1)}$, $x^{(2)}$

$$\left|L^1_{N_1,N_2}F^2(x^{(1)};x^{(2)})-F(x^{(1)};x^{(2)})L^1_{N_1,N_2}F(x^{(1)};x^{(2)})\right|\leq\frac{4\alpha_{12}\,\|\psi_1\|_C^2}{N_1}+\frac{4\alpha_{21}\,\|\psi_2\|_C^2}{N_2}. \tag{9}$$

In fact

$$\begin{aligned} F^2\left(\left(x^{(1)};x^{(2)}\right)_{i\to j}\right)-F^2\left(x^{(1)};x^{(2)}\right) &= \left(F\left(\left(x^{(1)};x^{(2)}\right)_{i\to j}\right)-F\left(x^{(1)};x^{(2)}\right)\right)\times \\ &\times\left(2F\left(x^{(1)};x^{(2)}\right)+\frac{1}{N_1}\left(\psi_1\left(x_j^{(2)}\right)-\psi_1\left(x_i^{(1)}\right)\right)\right) \\ &= 2F\left(x^{(1)};x^{(2)}\right)\left[F\left(\left(x^{(1)};x^{(2)}\right)_{i\to j}\right)-F\left(x^{(1)};x^{(2)}\right)\right]+ \\ &+\left[\frac{1}{N_1}\left(\psi_1\left(x_j^{(2)}\right)-\psi_1\left(x_i^{(1)}\right)\right)\right]^2. \end{aligned}$$

and similarly for expressions with $\left(x^{(1)};x^{(2)}\right)_{i\leftarrow j}$. Thus

$$L^1_{N_1,N_2}F^2(x^{(1)};x^{(2)}) = 2F\left(x^{(1)};x^{(2)}\right)\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{\alpha_{12}}{N_2}\cdot\left[F\left(\left(x^{(1)};x^{(2)}\right)_{i\to j}\right)-F\left(x^{(1)};x^{(2)}\right)\right]$$

$$
\begin{aligned}
& +\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{\alpha_{12}}{N_2}\cdot\left[\frac{1}{N_1}\left(\psi_1\left(x_j^{(2)}\right)-\psi_1\left(x_i^{(1)}\right)\right)\right]^2 \\
& +2F\left(x^{(1)};x^{(2)}\right)\sum_{j=1}^{N_2}\sum_{i=1}^{N_1}\frac{\alpha_{21}}{N_1}\cdot\left[F\left(\left(x^{(1)};x^{(2)}\right)_{i\leftarrow j}\right)-F\left(x^{(1)};x^{(2)}\right)\right] \\
& +\sum_{j=1}^{N_2}\sum_{i=1}^{N_1}\frac{\alpha_{21}}{N_1}\cdot\left[\frac{1}{N_2}\left(\psi_2\left(x_i^{(1)}\right)-\psi_2\left(x_j^{(2)}\right)\right)\right]^2 \\
= \quad & 2FL^1_{N_1,N_2}F+\frac{\alpha_{12}}{N_1}\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}\frac{1}{N_2N_1}\left(\psi_1\left(x_j^{(2)}\right)-\psi_1\left(x_i^{(1)}\right)\right)^2- \\
& +\frac{\alpha_{21}}{N_2}\sum_{j=1}^{N_2}\sum_{i=1}^{N_1}\frac{1}{N_1N_2}\left(\psi_2\left(x_i^{(1)}\right)-\psi_2\left(x_j^{(2)}\right)\right)^2.
\end{aligned}
$$

the estimate (9) follows from this. Lemma is proved.

Corollary 9

$$\sup_{t\le T}\mathsf{E}\left(W_{\psi,N_1,N_2}(t)\right)^2\to 0,\qquad N_1,N_2\to\infty.$$

Proof. As V_{ψ,N_1,N_2} is a martingale with mean zero, it is sufficient to prove that the expectation of

$$\int_0^t\left[L_{N_1,N_2}F^2\left(x^{(1)}(s);x^{(2)}(s)\right)-2F\left(x^{(1)}(s);x^{(2)}(s)\right)L_{N_1,N_2}F\left(x^{(1)}(s);x^{(2)}(s)\right)\right]ds$$

tends to zero. This follows from the estimate (7) of the lemma.

Proposition 10 *The sequence of distributions of real valued random processes*

$$F_{\psi,N_1,N_2}\left(x^{(1)}(t);x^{(2)}(t)\right),\ t\in[0,T],$$

is tight.

Proof of Proposition 10. Remind that the following representation holds

$$
\begin{aligned}
F_{\psi,N_1,N_2}\left(x^{(1)}(t);x^{(2)}(t)\right) \quad = \quad & F_{\psi,N_1,N_2}\left(x^{(1)}(0);x^{(2)}(0)\right)+W_{\psi,N_1,N_2}(t)+ \\
& +\int_0^t L_{N_1,N_2}F_{\psi,N_1,N_2}\left(x^{(1)}(s);x^{(2)}(s)\right)ds
\end{aligned}
$$

Note that our initial assumption is that the sequence $F_{\psi,N_1,N_2}\left(x^{(1)}(0);x^{(2)}(0)\right)$ weakly converges as $N_1,N_2\to\infty$.

Prove now that the sequence

$$\left\{\eta^{N_1,N_2}(t) = \int_0^t L_{N_1,N_2} F\left(x^{(1)}(s); x^{(2)}(s)\right) ds,\ t \in [0,T]\right\}_{N_1,N_2}.$$

is tight. We use subsection 6.1 of the Appendix. By assertion **i)** of the lemma

$$\left|\int_0^t L_{N_1,N_2} F\left(x^{(1)}(s); x^{(2)}(s)\right) ds\right| \le C_1(\psi, v_1, v_2, \alpha_{12}, \alpha_{21}) \cdot T\,,$$

thus, the condition 1) of the Appendix holds. The condition 2) also holds, as one can prove that

$$w'(\eta^{N_1,N_2}, \gamma) \le 2\gamma \cdot C_1(\psi, v_1, v_2, \alpha_{12}, \alpha_{21})\,.$$

Prove that the sequence $\{W_{\psi,N_1,N_2}(t),\ t \in [0,T]\}_{N_1,N_2}$ is tight. Using Kolmogorov's inequality for submartingales with right continuous trajectories (see [2]), we have the following estimate, uniform in N_1, N_2,

$$P\left(\sup_{t\le T} |W_{\psi,N_1,N_2}(t)| > C\right) \le \frac{\sup_{t\le T} \mathsf{E}\left(W_{\psi,N_1,N_2}(t)\right)^2}{C^2}$$

Then from the corollary 9 the condition 1) of Appendix holds. Thus

$$\begin{aligned}
P\left(|W_{\psi,N_1,N_2}(\tau+\theta) - W_{\psi,N_1,N_2}(\tau)| > \varepsilon\right) &\le \frac{\mathsf{E}\left(W_{\psi,N_1,N_2}(\tau+\theta) - W_{\psi,N_1,N_2}(\tau)\right)^2}{\varepsilon^2} = \\
&= \frac{\mathsf{E}\int_\tau^{\tau+\theta} V_{\psi,N_1,N_2}(s)\, ds}{\varepsilon^2} \le \\
&\le \frac{\theta \cdot (C_{12}(\alpha_{12}, \psi_1)/N_1 + C_{21}(\alpha_{21}, \psi_2)/N_2)}{\varepsilon^2}
\end{aligned}$$

Using this estimate one can check the sufficient condition of Aldous. Then Proposition 10 is proved.

This concludes also the proof of Proposition 6.

4.1.2 Weak solutions

Definition 11 *We say that the pair of functions $M(t) = (m_1(t,x), m_2(t,x))$ is a weak solution of the system (1)-(2), if for any pair $\phi_1(x), \phi_2(x) \in S(R)$ the following identities hold*

$$\begin{aligned}
\langle M(t), \phi\rangle &= \langle M(0), \phi\rangle + \\
&+ \int_0^t \langle M(s), (v_1\phi_1' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2, v_2\phi_2' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2)\rangle\, ds,
\end{aligned}$$

where $\phi(x) = (\phi_1(x), \phi_2(x))$, *and the action of* $G(x) = (g_1(x), g_2(x)$ *on the test function* $\phi(x)$ *can be written as*

$$\langle G, \phi \rangle = \int g_1(x)\phi_1(x)\,dx + \int g_2(x)\phi_2(x)\,dx$$

Note that from the representation (6) and the identity (8) it follows that

$$\begin{aligned}\langle M_{N_1,N_2}(t), \phi \rangle &= W_{\phi,N_1,N_2}(t) + \langle M_{N_1,N_2}(0), \phi \rangle + \\ &+ \int_0^t \langle M_{N_1,N_2}(s), (v_1\phi_1' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2, v_2\phi_2' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2) \rangle \, ds,\end{aligned}$$

Let $h = h(t) \in \Pi^T = D([0,T], S'(R) \times S'(R))$. For fixed ϕ define the functional

$$J_{\phi,T}(h) = \sup_{t \le T} \left| \langle h(t), \phi \rangle - \langle h(0), \phi \rangle - \int_0^t \langle h(s), (v_1\phi_1' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2, v_2\phi_2' - \alpha_{12}\phi_1 + \alpha_{21}\phi_2) \rangle \, ds \right|.$$

In particular,

$$\sup_{t \le T} |W_{\phi,N_1,N_2}(t)| = J_{\phi,T}(M_{N_1,N_2}).$$

The rest of the proof is standard (see [5]) and consists of three steps.

Step 1. From the definition of topology on Π^T it follows that $J_{\phi,T}(\cdot) : \Pi^T \to R_+$ is a continuous functional.

Step 2. Note that

$$\forall \varepsilon > 0 \quad P\left\{J_{\phi,T}(M_{N_1,N_2}) > \varepsilon\right\} \equiv P^T_{N_1,N_2}\left\{h : J_{\phi,T}(h) > \varepsilon\right\} \to 0 \qquad (N_1, N_2 \to \infty)$$

by Kolmogorov inequality and Corollary 9.

Step 3. As $J_{\phi,T}(\cdot)$ is continuous, then the set $\{h : J_{\phi,T}(h) > 0\}$ is open in Π^T. It follows now that for any limiting point P^T_∞ of the family $\left\{P^T_{N_1,N_2}\right\}_{N_1,N_2}$ we have

$$P^T_\infty\left\{h : J_{\phi,T}(h) > \varepsilon\right\} \le \limsup_{N_1,N_2} P^T_{N_1,N_2}\left\{h : J_{\phi,T}(h) > \varepsilon\right\}.$$

That is, for any $\varepsilon > 0$ we have $P^T_\infty\left\{h : J_{\phi,T}(h) > \varepsilon\right\} = 0$. In other words, all limiting points P^T_∞ of the family $\left\{P^T_{N_1,N_2}\right\}_{N_1,N_2}$ have support on the set $\{h : J_{\phi,T}(h) = 0\}$, which consists of weak solutions of (1)-(2).

This completes proof of Proposition 7.

The problem of uniqueness of the weak solution of (1)-(2) is quite simple because the system (1)-(2) is *linear*. In the Subsection 4.2 we shall see that this system of first order differential equations has a unique classical solution which can be obtained in explicit way.

4.2 Time asymptotics for the continuous model

We prove here Theorem 3.

Define the means (mass centrum) $a_i(t) = \int x m_i(t,x)\,dx$ and variance (momentum of inertia) $d_i(t) = \int (x - a_i(t))^2 m_i(t,x)\,dx$.

From (1)–(2) we get the following equations for the means

$$\begin{aligned} \dot{a_1} &= v_1 + \alpha_{12}\,(a_2 - a_1) \\ \dot{a_2} &= v_2 + \alpha_{21}\,(a_1 - a_2) \end{aligned}$$

It follows that equation for $a_2(t) - a_1(t)$ is closed and has the following solution

$$a_2(t) - a_1(t) = \frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}} \left(1 - e^{-(\alpha_{12}+\alpha_{21})t}\right) + (a_2(0) - a_1(0))\, e^{-(\alpha_{12}+\alpha_{21})t}.$$

Thus

$$a_2(t) - a_1(t) \to \frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}} \qquad (t \to +\infty)$$

and similarly

$$\frac{d}{dt} a_i(t) \to \frac{\alpha_{21} v_1 + \alpha_{12} v_2}{\alpha_{12} + \alpha_{21}} \qquad (t \to +\infty)$$

The equations for variances are

$$\begin{aligned} \dot{d_1} &= \alpha_{12}\,(d_2 - d_1) + \alpha_{12}\,(a_2(t) - a_1(t))^2 \\ \dot{d_2} &= \alpha_{21}\,(d_1 - d_2) + \alpha_{21}\,(a_1(t) - a_2(t))^2 \end{aligned}$$

Or, equivalently

$$\begin{aligned} \frac{d}{dt}\,(\alpha_{21} d_1 + \alpha_{12} d_2) &= 2\alpha_{12}\alpha_{21}\,(a_2(t) - a_1(t))^2 \\ \frac{d}{dt}\,(d_2 - d_1) &= -\,(\alpha_{12} + \alpha_{21})\,(d_2 - d_1) + (\alpha_{21} - \alpha_{12})\,(a_2(t) - a_1(t))^2 \end{aligned}$$

From this we get

$$d_2(t) - d_1(t) \to const = \left(\frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}}\right)^2 \cdot \frac{\alpha_{21} - \alpha_{12}}{\alpha_{12} + \alpha_{21}}$$

and

$$\frac{d}{dt}\,(\alpha_{21} d_1 + \alpha_{12} d_2) \to 2\alpha_{12}\alpha_{21} \left(\frac{v_2 - v_1}{\alpha_{12} + \alpha_{21}}\right)^2$$

Thus the growth of variances is asymptotically linear. Moreover, both are asymptotically equal.

Now we come to the solution of the equations. Define the Fourier transforms

$$m_i(x,t) = \int \exp(ixp) g_i(p,t) dp$$

We get

$$\frac{\partial g_1}{\partial t} + v_1 i p g_1 = \alpha_{12}(g_2 - g_1)$$

$$\frac{\partial g_2}{\partial t} + v_2 i p g_2 = \alpha_{21}(g_1 - g_2)$$

with initial conditions $m_i(0,x) = m_i(x)$, $i = 1,2$. We write this system in the vector form

$$\frac{dg}{dt} = Ag$$

where

$$A = \begin{pmatrix} -iv_1 p - \alpha_{12} & \alpha_{12} \\ \alpha_{21} & -iv_2 p - \alpha_{21} \end{pmatrix}$$

For eigenvalues we have

$$\lambda_{\pm} = -\frac{a}{2} \pm \sqrt{\frac{a^2}{4} - b}$$

where

$$a = i(v_1 + v_2)p + \alpha_{12} + \alpha_{21}, \qquad b = -v_1 v_2 p^2 + ip(v_1 \alpha_{21} + v_2 \alpha_{12})$$

One can write the solution as

$$g = C_+ \phi_+ \exp(t\lambda_+) + C_- \phi_- \exp(t\lambda_-)$$

where $\phi_{\pm}$ are eigenfunctions. Note that for small p there are two roots. One has $Re\lambda_- < 0$, thus strongly decreasing term. Another is

$$\lambda_+ = c_1 p + c_2 p^2 + O(p^3), c_2 \neq 0 \tag{10}$$

for small p.

Let ξ_t be a random variable with density $m(x,t)$, $g(k)$ - its characteristic function. We are interested in $\frac{1}{\sqrt{t}}(\xi_t - a), a = E\xi_t$, its characteristic function is

$$\exp(-ia\frac{k}{\sqrt{t}}) g(\frac{k}{\sqrt{t}})$$

Using (10) we get the result.

Remark 12 *One can see that there is no solution of the type*

$$m_i(t,x) = f_i(x - vt)$$

as then f_i would be exponents.

Remark 13 *For the singular initial conditions, that is when $x_k^{(i)}(0) = 0$ for $k = 1, ..., N_i; i = 1, 2$, one can get the same asymptotic results.*

5 Uniform estimates

To prove theorems 4 and 5 we will use an embedded Markov chain.

5.1 Embedded Markov chain

Consider the continuous time Markov chain $\xi_{N_1,N_2}(t) = \xi_{N_1,N_2}(t,\omega)$, defined by (5), and let

$$\tau_1(\omega) < \tau_2(\omega) < ... < \tau_n = \tau_n(\omega) < ...$$

be the random moments of particle jumps. Then $\tau_{n+1} - \tau_n$ are i. i. d. random variables exponentially distributed with mean $\gamma_{N_1,N_2} = (N_1\alpha_{12} + N_2\alpha_{21})^{-1}$.

We introduce a *discrete time* Markov chain $\zeta_{N_1,N_2}(n)$, $n = 1, 2, \dots$,

$$\zeta_{N_1,N_2}(n,\omega) = \xi_{N_1,N_2}(\tau_n(\omega),\omega)$$

with the same state space $R^{N_1+N_2}$. The idea is that the asymptotic behavior of the continuous time particle system $\xi_{N_1,N_2}(t)$ can be reduced to the asymptotic properties of the discrete time chain $\zeta_{N_1,N_2}(n)$. Indeed, by the Law of Large Numbers

$$\tau_n \sim \frac{n}{N_1\alpha_{12} + N_2\alpha_{21}} \qquad (n \to \infty)$$

In other words, if n is large, the value $n\gamma_{N_1,N_2} = n\left(N_1\alpha_{12} + N_2\alpha_{21}\right)^{-1}$ is asymptotically equal to the "physical" time t associated with the continuous time particle system $\xi_{N_1,N_2}(t)$. Similarly to the empirical mean and empirical variance we introduce, for the embedded chain,

$$X_i(n) = \frac{1}{N_i}\sum_{k=1}^{N_i} x_k^{(i)}(\tau_n), \qquad D_i(n) = \frac{1}{N_i}\sum_{k=1}^{N_i}\left(x_k^{(i)}(\tau_n) - X_i(n)\right)^2$$

Evidently, $X_i(n) = \overline{x^{(i)}}(\tau_n)$ and $D_i(n) = S_i^2(\tau_n)$. In the sequel we shall deal with their expected values

$$\mu_i(n) = \mathsf{E}X_i(n), \qquad d_i(n) = \mathsf{E}D_i(n),$$

and shall need also the notation

$$l_{12}(n) = \mu_1(n) - \mu_2(n), \qquad r(n) = \mathsf{E}\left(X_1(n) - X_2(n)\right)^2$$

5.2 Empirical means

Here we prove Theorem 4.

The following lemma can be checked by a straightforward calculation.

Lemma 14 *The functions $\mu_i(n)$ satisfy to the following closed system*

$$\begin{aligned}\mu_1(n+1) &= \mu_1(n) + [\alpha_{12}(\mu_2(n) - \mu_1(n)) + v_1]\,\gamma_{N_1,N_2} + \alpha_{12}(v_2 - v_1)\,\gamma^2_{N_1,N_2} \\ \mu_2(n+1) &= \mu_2(n) + [\alpha_{21}(\mu_1(n) - \mu_2(n)) + v_2]\,\gamma_{N_1,N_2} + \alpha_{21}(v_1 - v_2)\,\gamma^2_{N_1,N_2}\end{aligned}$$

For l_{12} the equation is also linear and closed. Namely,

$$\begin{aligned}l_{12}(n+1) &= l_{12}(n) + [-(\alpha_{12}+\alpha_{21})\,l_{12}(n) + (v_1 - v_2)]\,\gamma_{N_1,N_2} + (\alpha_{12}+\alpha_{21})(v_2 - v_1)\,\gamma^2_{N_1,N_2} \\ &= l_{12}(n)\,[1 - \gamma_{N_1,N_2}(\alpha_{12}+\alpha_{21})] + \gamma_{N_1,N_2}(v_1 - v_2)\,[1 - \gamma_{N_1,N_2}(\alpha_{12}+\alpha_{21})],\end{aligned}$$

and thus we get

$$\begin{aligned}l_{12}(n) &= l_{12}(0)R^n + \frac{v_1 - v_2}{\alpha_{12}+\alpha_{21}}(1 - R^n)\,R, &(11)\\ &= \frac{v_1 - v_2}{\alpha_{12}+\alpha_{21}}R + \left(l_{12}(0) - \frac{v_1 - v_2}{\alpha_{12}+\alpha_{21}}R\right)R^n &(12)\end{aligned}$$

where $R = 1 - \gamma_{N_1,N_2}(\alpha_{12}+\alpha_{21})$.

The variables $S(n) = \alpha_{21}\mu_1(n) + \alpha_{12}\mu_2(n)$ also satisfy the recurrent equations

$$S(n+1) = S(n) + \gamma_{N_1,N_2}\,[\alpha_{21}v_1 + \alpha_{12}v_2]$$

Thus

$$S(n) = S(0) + n\frac{\alpha_{21}v_1 + \alpha_{12}v_2}{N_1\alpha_{12} + N_2\alpha_{21}} = S(0) + (n\gamma_{N_1,N_2})\cdot(\alpha_{21}v_1 + \alpha_{12}v_2) \qquad (13)$$

From (12) and (13) the statement of Theorem 4 follows.

5.3 Empirical variances

Closed equations We would like to get close recurrent equations for $d_i(n)$.

Lemma 15 *The following identity holds*

$$\mathsf{E}\left(D_1(n+1)\,\Big|\,\left(x_i^{(1)}(t), x_j^{(2)}(t),\, i = 1,\ldots,N_1,\, j = 1,\ldots,N_2\right),\, t \le \tau_n\right) =$$

$$= D_1(n) + \alpha_{12}\gamma_{N_1,N_2}\left[\frac{N_1 - 1}{N_1}\left(D_2(n) - D_1(n) + (X_1(n) - X_2(n))^2\right) - \frac{2}{N_1}D_1(n) + \right.$$

$$\left. +2\,\frac{N_1 - 1}{N_1}\left(\gamma_{N_1,N_2}(v_1 - v_2)(X_1(n) - X_2(n)) + \gamma^2_{N_1,N_2}(v_1 - v_2)^2\right)\right]$$

and a similar identity for $\mathsf{E}(D_2(n+1)\,|\,\ldots)$ can be obtained by a simple exchange of indices $1 \leftrightarrow 2$.

Proof of this lemma is a straightforward calculation. Taking expectations in the above formulae we see that in the equations for $d_i(n)$ the term $r(n)$ is also involved.

Consider the vector $w(n) = (d_1(n), d_2(n), r(n))^T$. We have

$$w(n+1) = Aw(n) + f(n) + g, \tag{14}$$

where A is a (3×3)-matrix, not depending on n, $f(n)$ is a bounded vector function of n, g is a constant vector

$$A = E + \gamma_{N_1,N_2} B, \tag{15}$$

$$B = B_1 + B_2 = \begin{pmatrix} -\alpha_{12} & \alpha_{12} & \alpha_{12} \\ \alpha_{21} & -\alpha_{21} & \alpha_{21} \\ 0 & 0 & -2(\alpha_{12}+\alpha_{21}) \end{pmatrix} + \begin{pmatrix} -\alpha_{12}/N_1 & -\alpha_{12}/N_1 & -\alpha_{12}/N_1 \\ -\alpha_{21}/N_2 & -\alpha_{21}/N_2 & -\alpha_{21}/N_2 \\ \dfrac{\alpha_{12}}{N_1} + \dfrac{\alpha_{21}}{N_2} & \dfrac{\alpha_{12}}{N_1} + \dfrac{\alpha_{21}}{N_2} & \dfrac{\alpha_{12}}{N_1} + \dfrac{\alpha_{21}}{N_2} \end{pmatrix},$$

$$f(n) = 2\,\gamma_{N_1,N_2}\,(v_1 - v_2)\,l_{12}(n)\,\vec{q}_{N_1,N_2}, \quad \vec{q}_{N_1,N_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + \gamma_{N_1,N_2} \begin{pmatrix} \alpha_{12} - \alpha_{12}/N_1 \\ \alpha_{21} - \alpha_{21}/N_2 \\ -(\alpha_{12}+\alpha_{21}) \end{pmatrix}, \tag{16}$$

$$g = \gamma_{N_1,N_2}^2 (v_1 - v_2)^2 \begin{pmatrix} \alpha_{12}\gamma_{N_1,N_2} \\ \alpha_{21}\gamma_{N_1,N_2} \\ 2 \end{pmatrix}.$$

Note that $A, f(n), g$ depend on N_1, N_2 and on other parameters of the model. Note that for sufficiently large N_1, N_2 (if α_{ij} are fixed) all components of the vector $\vec{q}_{N_1,N_2}$ are positive. Then g, the coordinates of f and the matrix elements of A are all positive. Thus Perron-Frobenius theory is applicable.

Spectral properties of the matrix A It is easy to check that the matrix B_1 has three distinct eigenvalues $\lambda_1 = -(\alpha_{12}+\alpha_{21})$, $\lambda_2 = 0$, $\lambda_3 = -2(\alpha_{12}+\alpha_{21})$. We will study asymptotics for the case when $N_1 = c_1 N$, $N_2 = c_2 N$ and $N \to \infty$. Denote $\Delta := c_1\alpha_{12} + c_2\alpha_{21}$, that is $\gamma_{N_1,N_2} = (N\Delta)^{-1}$.

For large N_i the matrix B is a small perturbation of the matrix B_1

$$B = B_1 + \frac{1}{N} B_{2,k} = B_1 + \frac{1}{N} \begin{pmatrix} -\alpha_{12}/c_1 & -\alpha_{12}/c_1 & -\alpha_{12}/c_1 \\ -\alpha_{21}/c_2 & -\alpha_{21}/c_2 & -\alpha_{21}/c_2 \\ \dfrac{\alpha_{12}}{c_1} + \dfrac{\alpha_{21}}{c_2} & \dfrac{\alpha_{12}}{c_1} + \dfrac{\alpha_{21}}{c_2} & \dfrac{\alpha_{12}}{c_1} + \dfrac{\alpha_{21}}{c_2} \end{pmatrix}.$$

We will use perturbation theory to get eigenvalues of B. For $\lambda_1(N)$ and $\lambda_3(N)$ it is sufficient to write

$$\lambda_1(N) = -(\alpha_{12}+\alpha_{21}) + \underline{O}\left(\frac{1}{N}\right), \tag{17}$$

$$\lambda_3(N) = -2(\alpha_{12}+\alpha_{21}) + \underline{O}\left(\frac{1}{N}\right), \tag{18}$$

however for $\lambda_2(N)$ we will use the result from [4], that

$$\lambda_2(N) = \frac{1}{N}\left(\psi' B_{2,k}\phi\right) + \underline{O}\left(\frac{1}{N^2}\right), \tag{19}$$

where the column vector ϕ is the right eigenvector of B_1 with eigenvalue 0, the row vector ψ' is the left eigenvector of B_1 with eigenvalue 0, and moreover $\psi'\phi = 1$. One can take

$$\phi = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \qquad \psi' = \left(1 + \frac{\alpha_{21}}{\alpha_{12}}, 1 + \frac{\alpha_{12}}{\alpha_{21}}, 1\right)/Z, \qquad Z = (\alpha_{12} + \alpha_{21})\left(\frac{1}{\alpha_{12}} + \frac{1}{\alpha_{21}}\right).$$

Substituting these values to (19), we get

$$\lambda_2(N) = -\frac{\varkappa_2}{N} + \underline{O}\left(\frac{1}{N^2}\right), \tag{20}$$

where

$$\varkappa_2 = 2Z^{-1}\left(\frac{\alpha_{21}}{c_1} + \frac{\alpha_{12}}{c_2}\right) \tag{21}$$

Denote $\sigma_1(N)$, $\sigma_1(N)$, $\sigma_1(N)$ the eigenvalues of the matrix A. From (15) and (17)–(19) we have the following assertion.

Lemma 16 *The eigenvalues of A are*

$$\begin{aligned} \sigma_1(N) &= 1 - \frac{b_1}{N} + \underline{O}\left(\frac{1}{N^2}\right) \\ \sigma_2(N) &= 1 - \frac{b_2}{N^2} + \underline{O}\left(\frac{1}{N^3}\right) \\ \sigma_3(N) &= 1 - \frac{b_3}{N} + \underline{O}\left(\frac{1}{N^2}\right), \end{aligned} \tag{22}$$

for some positive constants b_1, b_2, b_3.

We will need also the eigenvectors of A, whiçch we denote correspondingly by $e_1^{(N)}, e_2^{(N)}, e_3^{(N)}$. It is clear that they are also the eigenvectors of the matrix $B_1 + \frac{1}{N}B_{2,k}$. We have that $e_1^{(N)}, e_2^{(N)}, e_3^{(N)}$ are small perturbations of the eigenvectors e_1, e_2, e_3 of the matrix B_1. Again using the perturbation theory [4] we have

$$e_1^{(N)} = \begin{pmatrix} -\alpha_{12} \\ \alpha_{21} \\ 0 \end{pmatrix} + \underline{O}\left(\frac{1}{N}\right), \quad e_2^{(N)} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + \underline{O}\left(\frac{1}{N}\right), \quad e_3^{(N)} = \begin{pmatrix} -\alpha_{12}^2 \\ -\alpha_{21}^2 \\ (\alpha_{12} + \alpha_{21})^2 \end{pmatrix} + \underline{O}\left(\frac{1}{N}\right).$$

It is clear that with (17), (18) i (20) it is not difficult to find explicitely the constants b_i. We will need only b_2:

$$b_2 = \frac{\varkappa_2}{\Delta} = \frac{2}{Z} \cdot \frac{\dfrac{\alpha_{12}}{c_1} + \dfrac{\alpha_{21}}{c_2}}{c_1\alpha_{12} + c_2\alpha_{21}}. \tag{23}$$

Some lemmas on the asymptotic behaviour The solution of the equation (14) can be uniquely written as

$$w(n) = A^n w(0) + \sum_{j=1}^{n} A^{j-1} f(n-j) + (1-A)^{-1}(1-A^n)g. \tag{24}$$

The following result shows that the first and last tems in (24) do not influence the asymptotics of $w(n)$.

Lemma 17 *The following estimates hold uniformy in n and N*

$$\|A^n w(0)\| \le Const,$$
$$\left\|(1-A)^{-1}(1-A^n)g\right\| \le Const.$$

Proof. Using the basis of eigenvectors of A we can write

$$w(0) = \sum_{i=1}^{3} k_{w,i} e_i^{(N)}$$

Then

$$A^n w(0) = \sum_{i=1}^{3} k_{w,i} \left(\sigma_i(N)\right)^n e_i^{(N)}.$$

Note that $\sup_N \left\|e_i^{(N)}\right\| < \infty$. Moreover, $|\sigma_i(N)| < 1$, starting with some N. Then the first estimate follows. To get the second estimate we write

$$g = \gamma^2_{N_1,N_2} \sum_{i=1}^{3} k^{\circ}_{g,i}(N) e_i^{(N)}$$

with the coefficients $k^{\circ}_{g,i}(N)$, bounded in N. Apply the operator $(1-A)^{-1}(1-A^n)$ to the latter expansion and note that by Lemma 16

$$\gamma^2_{N_1,N_2} \frac{1-(\sigma_i(N))^n}{1-\sigma_i(N)} \le \frac{Const}{N} \qquad i = 1, 3,$$

and

$$\gamma^2_{N_1,N_2} \frac{1-(\sigma_2(N))^n}{1-\sigma_2(N)} \le Const$$

Then we get the estimate. ■

Now we will analyze the second term in (24)

$$V_N(n) := \sum_{j=1}^{n} A^{j-1} f(n-j),$$

Note that the vector function $f(n)$, defined by the formula (16), is known explicitely with the formula (11).

Let $\xi_1, \xi_2, \xi_3 \in \mathbb{R}$ are such that

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \sum_{i=1}^{3} \xi_i e_i .$$

Then we have immeditely that

$$\xi_1 = \frac{\alpha_{21} - \alpha_{12}}{(\alpha_{12} + \alpha_{21})^2}, \quad \xi_2 = \frac{\alpha_{12}\alpha_{21}}{(\alpha_{12} + \alpha_{21})^2}, \quad \xi_3 = \frac{1}{(\alpha_{12} + \alpha_{21})^2}.$$

If $\xi_i(N) \in \mathbb{R}$ is the coefficient in the expansion $\overrightarrow{q}_{N_1,N_2} = \sum_{i=1}^{3} \xi_i(N) e_i^{(N)}$, then obviously

$$\xi_i(N) = \xi_i + \underline{O}\left(\frac{1}{N}\right), \quad i = 1, 2, 3. \tag{25}$$

We have then

$$V_N(n) = \sum_{i=1}^{3} \xi_i(N) \left(2\,\gamma_{N_1,N_2}\,(v_1 - v_2) \sum_{j=1}^{n} l_{12}(n-j)\,(\sigma_i(N))^{j-1} \right) e_i^{(N)}.$$

By (25), and neglecting $\underline{O}\left(\frac{1}{N}\right)$, we have that for $N \to \infty$ the asymptotics of $V_N(n)$ coincides with the asymptotics of the sum

$$V_N^1(n) := \sum_{i=1}^{3} \xi_i \left(2\,\gamma_{N_1,N_2}\,(v_1 - v_2) \sum_{j=1}^{n} l_{12}(n-j)\,(\sigma_i(N))^{j-1} \right) e_i^{(N)}. \tag{26}$$

Lemma 18 *For $n = N\theta(N)$, where $\theta(N) \to +\infty$, the asymptotics of $V_N^1(n)$ is defined by the second term, that is the first and third are small with respect to the second.*

Remind that $\sigma_2(N)$ is the maximal eigenvalue of the positive matrix A. Then this corresponds to Perron-Frobenius theory.

Proof. Consider the formuma (11). If we assume, that $l_{12}(0) < 0$, then from $v_1 < v_2$ it follows that $f(n) > 0$ for all n (we use it below). Moreover, from (11) one can get that there exist constants $C_1 > C_2 > 0$, which do not depend on N and n, such that

$$0 < C_2 < (v_1 - v_2) l_{12}(n) < C_1 \quad \forall n, N \tag{27}$$

Thus, the coefficient of $e_2^{(N)}$ in the sum (26) is positive and can be estimated from below as

$$2\xi_2 C_2\,\gamma_{N_1,N_2} \sum_{j=1}^{n} (\sigma_2(N))^{j-1} = 2\xi_2 C_2\,\gamma_{N_1,N_2} \frac{1 - (\sigma_2(N))^n}{1 - \sigma_2(N)}$$

Similarly, for $i = 1, 3$ the absolute values of the coefficients of $e_i^{(N)}$ in the sum (26) can be estimated from above as

$$2\xi_i C_1\, \gamma_{N_1,N_2} \sum_{j=1}^{n} (\sigma_i(N))^{j-1} = 2\xi_i C_1\, \gamma_{N_1,N_2} \frac{1-(\sigma_i(N))^n}{1-\sigma_i(N)}$$

Thus, to end the proof of the lemma it is sufficient to compare the asymptotics of the following three functions

$$\frac{1-(\sigma_1(N))^{N\theta(N)}}{1-\sigma_1(N)}, \quad \frac{1-(\sigma_2(N))^{N\theta(N)}}{1-\sigma_2(N)}, \quad \frac{1-(\sigma_3(N))^{N\theta(N)}}{1-\sigma_3(N)}$$

and to show that the first and the third are small with respect to the second. It is convenient to consider separately two cases: **a)** $\theta(N) \to \infty$, $\theta(N)/N \to 0$, **b)** $\theta(N) \geq cN$, and use Lemma 16. We omit these details. ■

Asymptotic behavior of expectations of empirical variances

Consider now the asymptotics for the means

$$R_1(t) = \mathsf{E}S_1^2(t) = \frac{1}{N_1} \sum_{i=1}^{N_1} \left(x_i^{(1)}\right)^2 (t) - \left(\overline{x^{(1)}}\right)^2 (t)$$

and

$$R_2(t) = \mathsf{E}S_2^2(t) = \frac{1}{N_2} \sum_{j=1}^{N_2} \left(x_j^{(2)}\right)^2 (t) - \left(\overline{x^{(2)}}\right)^2 (t).$$

Proof of Theorem 5. Remind that the intervals between jumps of the embedded chain have exponential distribution with the mean $\gamma_{N_1,N_2} = (N\Delta)^{-1}$, then for large N the connection between discrete time n of the embedded chain and absolute time t is

$$n \sim t/\gamma_{N_1,N_2} = (N\Delta)t$$

Thus we can take $d_i(\,(N\Delta)t\,)$ instead of $R_i(t)$. Remind also that d_1 and d_2 are the first and second components of the vector w correspondingly.

Now we can use the lemma 18, which shows that, as $t(N) \to \infty$, the asymptotics of $w(\,(N\Delta)t(N)\,)$ coincides with the asymptotics of the vector

$$\xi_2 \left(2\,\gamma_{N_1,N_2}\,(v_1 - v_2) \sum_{j=1}^{(N\Delta)t(N)} l_{12}(\,(N\Delta)t(N) - j\,)\,(\sigma_2(N))^{j-1} \right) e_2^{(N)}.$$

As $e_2^{(N)} = (1,1,0)^T + \underline{O}\left(N^{-1}\right)$, then

$$d_i(\,(N\Delta)t\,) \sim 2\,\xi_2\,\gamma_{N_1,N_2}\,(v_1 - v_2) \sum_{j=1}^{(N\Delta)t(N)} l_{12}(\,(N\Delta)t(N) - j\,)\,(\sigma_2(N))^{j-1}.$$

To find the asymptotics of this expression, we use the representation (12), which gives

$$l_{12}(n) = C'_{1,N} + C'_{2,N} R^n, \qquad C'_{1,N} \to \frac{v_1 - v_2}{\alpha_{12} + \alpha_{21}}, \quad C'_{2,N} \to C'_{2,\infty} \quad (N \to \infty).$$

Note that the following estimate holds uniformly in N and n

$$\begin{aligned}\left| \gamma_{N_1,N_2} \sum_{j=1}^{n} C'_{2,N} R^{n-j} \left(\sigma_2(N)\right)^{j-1} \right| &\le (N\Delta)^{-1} \left|C'_{2,N}\right| \sum_{j=1}^{\infty} R^{n-j} = (N\Delta)^{-1} \left|C'_{2,N}\right| \cdot \frac{1}{1-R} \\ &\le Const.\end{aligned}$$

Consider now the asymptotics of the following expression

$$2\,\xi_2\,\gamma_{N_1,N_2}\,(v_1 - v_2) \sum_{j=1}^{(N\Delta)t(N)} \frac{v_1 - v_2}{\alpha_{12} + \alpha_{21}} \left(\sigma_2(N)\right)^{j-1} = 2\,\xi_2 \frac{1}{N\Delta} \cdot \frac{(v_1 - v_2)^2}{\alpha_{12} + \alpha_{21}} \cdot \frac{1 - \left(\sigma_2(N)\right)^{(N\Delta)t(N)}}{1 - \sigma_2(N)}.$$

By (22) and (23)

$$\sigma_2(N) = 1 - \frac{(\varkappa_2/\Delta)}{N^2} + \underline{O}\left(\frac{1}{N^3}\right)$$

and thus the problem is reduced to the study of asymptotics of

$$\frac{2\,\xi_2\,(v_1 - v_2)^2}{\varkappa_2\,(\alpha_{12} + \alpha_{21})} N \left(1 - \left(1 - \frac{(\varkappa_2/\Delta)}{N^2}\right)^{(N\Delta)t(N)}\right).$$

Now the theorem easily follows.

6 Appendix

6.1 Probability measures on the Skorohod space: tightness

Let $\{(\xi_t^n, t \in [0,T])\}_{n\in\mathbf{N}}$ be a sequence of real random processes which trajectories are right-continuous and admit left-hand limits for every $0 < t \le T$. We will consider ξ^n as random elements with values in the Skorohod space $D_T(R) := D\left([0,T], R^1\right)$ with the standard topology. Denote P_T^n the distribution of ξ^n, defined on the measurable space $(D_T(R), \mathcal{B}(D_T(R)))$. The following result can be found in [1].

Theorem 19 *The sequence of probability measures $\{P_T^n\}_{n\in\mathbf{N}}$ is tight iff the following two conditions hold:*

1) for any $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that

$$\sup_n P_T^n \left(\sup_{0\le t\le T} |\xi_t^n| > C(\varepsilon) \right) \le \varepsilon\,;$$

2) for any $\varepsilon > 0$

$$\lim_{\gamma \to 0} \limsup_n P_T^n \left(\xi . : \, w'(\xi; \gamma) > \varepsilon \right) = 0 \, ,$$

where for any function $f : [0, T] \to R$ *and any* $\gamma > 0$ *we define*

$$w'(f; \gamma) = \inf_{\{t_i\}_{i=1}^r} \max_{i<r} \sup_{t_i \leq s < t < t_{i+1}} |f(t) - f(s)| \, ,$$

moreover the inf *is over all partitions of the interval* $[0, T]$ *such that*

$$0 = t_0 < t_1 < \cdots < t_r = T, \qquad t_i - t_{i-1} > \gamma, \quad i = 1, \ldots, r.$$

The following theorem is known as the sufficient condition of Aldous [5].

Theorem 20 *Condition 2) of the previous theorem follows from the following condition*

$$\forall \varepsilon > 0 \qquad \lim_{\gamma \to 0} \limsup_n \sup_{\tau \in R_T, \, \theta \leq \gamma} P_T^n \left(|\xi_{\tau+\theta} - \xi_\tau| > \varepsilon \right) = 0 \, ,$$

where $\mathcal{R}_T$ *is the set of Markov moments (stopping times) not exceeding* T *.*

6.2 Strong topology on the Skorohod space. Mitoma theorem

Remind that Schwartz space $S(R)$ is a Frechet space (complete locally convex space, the topology of which is generated by countable family of seminorms, that implies metrizability, see [10]). In the dual space $S'(R)$ of tempered distributions there are at least two ways to define topology (both not metrizable):

1) *weak topology* on $S'(R)$, where all functionals

$$\langle \cdot \, , \phi \rangle \, , \quad \phi \in S(R)$$

are continuous.

2) *strong topology* on $S'(R)$, which is generated by the set of seminorms

$$\left\{ \rho_A(M) = \sup_{\phi \in A} |\langle M, \phi \rangle| \, : \, A \subset S(R) \, - \, bounded \right\}.$$

We shall consider $S'(R)$ as equipped with the strong topology. Details can be found in [10].

The problem of introducing of the Skorohod topology on the space $D_T(S') := D([0, T], S'(R))$ was studied in [8] and [11]. The topology on this space is defined as follows. Let $\{\rho_A\}$ be a family of seminorms, which generates strong topology in $S'(R)$. For each seminorm ρ_A define a pseudometrics

$$d_A(y, z) = \inf_{\lambda \in \Lambda} \left\{ \sup_t \left| y_t - z_{\lambda(t)} \right| + \sup_{t \neq s} \left| \log \frac{\lambda(t) - \lambda(s)}{t - s} \right| \right\}, \quad y, z \in D_T(S'),$$

where the inf is over the set $\Lambda = \{\lambda = \lambda(t),\, t \in [0,T]\}$ of all strictly increasing maps of the interval $[0,T]$ into itself. Equipped with the topology of the projective limit for the family $\{d_A\}$ the set $D_T(S')$ becomes a completely regular topological space.

Let $\mathcal{B}(D_T(S'))$ be the corresponding Borel σ-algebra. Let $\{P_n\}$ be a sequence of probability measures on $(D_T(S'), \mathcal{B}(D_T(S')))$. For each $\phi \in S(R)$ consider a map $\mathcal{I}_\phi : y \in D_T(S') \to y.(\phi) \in D_T(R)$. The following result belongs to I. Mitoma [8].

Theorem 21 *Suppose that for any $\phi \in S(R)$ the sequence $\left\{ P_n \mathcal{I}_\phi^{-1} \right\}$ is tight in in $D_T(R)$. Then the sequence $\{P_n\}$ itself is tight in $D_T(S')$.*

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